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Quasi-p-Large Subgroups of Abelian Groups

by

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In a monumental paper by R. S. Pierce ([3]) the notion of large subgroups was introduced in the theory of primary abelian groups and used to investigate their endomorphism rings. A subgroup L of a primary group G is said to be a *large subgroup* of G if L is fully-invariant in G and $G=B+L$ for all basic subgroups B of G. In [1] subgroups satisfying only the second condition were considered and called *quasi-large subgroups*. It was shown there that a subgroup of a primary group is quasi-large if and only if it contains a large-subgroup. In the present paper we consider the natural generalization of these concepts to arbitrary abelian groups, using the notion of p-basic subgroups. After some useful remarks on p-basic subgroups we characterize the quasi-p-large and p-large subgroups of torsion free groups and give some partial results on the mixed case. For notations and terminology we follow [2] except for items directly introduced here. The word "group" shall mean abelian group.

1. Some remarks on p-basic subgroups

For the definition and usual properties of p-basic subgroups we refer the reader to [2] Chapter VI, p. 135–158. We establish here an alternative way of defining p-basic subgroups as well as some further properties not contained in the reference above.

Notation. Let p be a fixed prime number, P the set of all prime numbers. Let G be a group. We denote by G_q the q-primary subgroup of G for each prime number q. Given $q\in P$ we let

$$G(q)=\bigoplus_{\substack{r\in P\\ r\neq q}} G_r\,,$$

$$G^q=G/G(q)$$

and for a subgroup A of G we let $A^q=(A+G(q))/G(q)$.

PROPOSITION 1.1. *Let $\{x_\lambda\}_{\lambda\in\Lambda}$ be a family of elements of a group G. If $\{x_\lambda+(G_p+pG)\}_{\lambda\in\Lambda}$ is independent in $G/(G_p+pG)$, then $\{x_\lambda\}_{\lambda\in\Lambda}$ is an independent set which generates a torsion free p-pure subgroup of G.*

Proof. Let $\sum a_\lambda x_\lambda = 0$, then the hypothesis implies that $a_\lambda = pa'_\lambda$ for each $\lambda \in \Lambda$. Therefore $p(\sum a'_\lambda x_\lambda) = 0$ and $\sum a'_\lambda x_\lambda + (G_p + pG) = 0$. This again implies that a'_λ is divisible by p. Repeating this argument we see that any power of p divides a_λ. Therefore $a_\lambda = 0$ for each $\lambda \in \Lambda$. Suppose now that $p^n g = \sum a_\lambda x_\lambda$ for some $n \in N$ and $g \in G$. $\sum a_\lambda x_\lambda + (G_p + pG) = 0$ and the same reasoning as above shows that $a_\lambda = pa'_\lambda$. Repeating this argument we conclude that each a_λ is divisible by p^n. Therefore the subgroup generated by the x_λ's is p-pure in G.

The next two propositions provide an alternate way of defining p-basic subgroups. The proofs are straightforward and are omitted here.

PROPOSITION 1.2. *Let $\{x_\lambda\}_{\lambda \in \Lambda}$ be a family of elements of G such that $\{x_\lambda + (G_p + pG)\}_{\lambda \in \Lambda}$ is a basis of $G/(G_p + pG)$, and let $\{y_\mu\}_{\mu \in M}$ be a basis of a basic subgroup of G_p, then $\{x_\lambda\}_{\lambda \in \Lambda} \cup \{y_\mu\}_{\mu \in M}$ is a basis of a p-basic subgroup of G.*

PROPOSITION 1.3. *Let B be a p-basic subgroup of G. Write $B = B_0 \oplus B_p$, where B_p is generated by the elements of prime power order and B_0 by those of infinite order in some basis of B. Let $\{x_\lambda\}_{\lambda \in \Lambda}$ be a basis of B_0, then $\{x_\lambda + (G_p + pG)\}_{\lambda \in \Lambda}$ is a basis of $G/(G_p + pG)$ and B_p is a basic subgroup of G_p.*

COROLLARY. *Let G be a group such that $G_p = 0$, then $\{x_\lambda\}_{\lambda \in \Lambda}$ is a basis of a p-basic subgroup of G if and only if $\{x_\lambda + pG\}_{\lambda \in \Lambda}$ is a basis of G/pG.*

Next we turn to a kind of localisation principle with respect to the primary components of a group.

PROPOSITION 1.4. *Let G be a group and $q \in P$. Every p-basic subgroup $\bar{B}$ of G^q is of the form B^q where B is a p-basic subgroup of G.*

Proof. For the meaning of G^q and B^q see notation in the beginning of this section. First we consider the case where $p \neq q$. Let $\bar{B} = B^*/G(q)$, where $B^* \supset G(q)$. Since $(G^q)_p = 0$, $\bar{B}$ is a free group therefore

$$B^* = G(q) \oplus \left(\bigoplus_{\lambda \in \Lambda} \langle x_\lambda \rangle \right).$$

Let

$$\bigoplus_{\lambda \in \Lambda} \langle x_\lambda \rangle = B_0 ,$$

then $G = B_0 \oplus G(q) + pG = B_0 + G_p + pG$, therefore $\{x_\lambda + (G_p + pG)\}$ is a basis of $G/(G_p + pG)$ and B_0 can be extended by B_p, a basic subgroup of G_p, to form a p-basic subgroup B of G (see Prop. 1.2). Thus $B^q = (B + G(q)/G(q) = (B_0 \oplus G(q))/G(q) = B^*/G(q) = \bar{B}$. Suppose now that $q = p$. Let $\bar{B} = B^*/G(p)$ where $B^* \supset G(p)$. Since $\bar{B}$ is a direct sum of cyclic groups and $G(p)$ is a pure subgroup of G we have: $B^* = B \oplus G(p)$. Now B is a direct sum of cyclic groups of infinite or prime power order which is clearly p-pure in G. We need only show that G/B is p-divisible, but $G^p/\bar{B} \cong G/B^*$ is p-divisible, thus $G/(B \oplus G(p))$ is p-divisible. That is to say $(B \oplus G(p)) +$

$pG=G$. However $G(p)\subset pG$ therefore $B+pG=G$ and G/B is p-divisible. Clearly $\bar{B}=B^p$.

PROPOSITION 1.5. *Let B be a p-basic subgroup of a group G, then for every $q\in P$, B^q is a p-basic subgroup of G^q.*

Proof. $B^q=(B+G(q))/G(q)$ therefore $G^q/B^q\cong G/(B+G(q))$ is a homomorphic image of G/B. Therefore, G^q/B^q is p-divisible. We need only show that $B+G(q)$ is a p-pure subgroup of G for any $q\in P$. If $p=q$ then $G(q)=G(p)$ is p-divisible and since B is p-pure in G, $B+G(q)$ is p-pure in G. If $p\neq q$, B/B_p is a p-basic subgroup of G/B_p and since G_p/B_p is divisible, we conclude that $B+G_p$ is p-pure in G. Now write

$$G(q)=G_p\oplus\Bigg(\bigoplus_{\substack{r\in P\\ r\neq p,\, r\neq q}} G_r\Bigg)$$

then $B+G(q)=(B+G_p)+G(q,p)$ where

$$G(q,p)=\bigoplus_{\substack{r\neq p\\ r\neq q}} G_r\,.$$

The left hand side term is p-pure in G and the right hand side is p-divisible. Therefore their sum is a p-pure subgroup of G.

2. p-Large and quasi-p-large subgroups

DEFINITION 2.1. Let p be a fixed prime number. A subgroup A of a group G is said to be *quasi-p-large* if $G=A+B$ for all p-basic subgroups B of G. It is said *p-large* if it is in addition a fully invariant subgroup of G.

PROPOSITION 2.2. *Let G be a group and A a subgroup of G. Then A is a quasi-p-large subgroup of G if and only if A^q is a quasi-p-large subgroup of G^q for all $q\in P$.*

Proof. Let A be quasi-p-large and let $q\in P$. Let $\bar{B}$ be a p-basic subgroup of G^q, then by Proposition 1.4, $\bar{B}=B^q$ for some p-basic subgroup B of G. Now $A+B=G$, therefore $A+G(q)+B+G(q)=G$, and $A^q+B^q=((A+G(q)/G(q))+((B+G(q))/G(q))=G/G(q)=G^q$. Therefore A^q is quasi-p-large in G^q. Conversely suppose that for all $q\in P$, A^q is quasi-p-large in G^q and let B be a p-basic subgroup of G. From Proposition 1.5, B^q is a p-basic subgroup of G^q for all $q\in P$. Therefore: $A^q+B^q=G^q$, that is to say for all $q\in P$,

$$(A+G(q))+(B+G(q))=G=A+B+G(q)\,.$$

Therefore $G/(A+B)\cong G(q)/G(q)\cap(A+B)$, for all $q\in P$. Clearly $G/(A+B)$ is a torsion group and $(G/A+B)_q=0$, for all $q\in P$. Therefore $G/(A+B)=0$, that is to say $G=A+B$, and A is a quasi-p-large subgroup of G.

In the next section we characterize quasi-p-large subgroups of *p-torsion free* groups i.e. groups G which have trivial p-primary component. But before, we state here a fact that is easy to prove:

LEMMA 2.3. *Let B be a p-basic subgroup of a group G. Then qB is a p-basic subgroup of G for all $q \neq p$, $q \in P$.*

3. Quasi-p-large subgroups of p-torsion free groups

This is, as it turns out, the easy case. We need a preparatory lemma. All groups here are assumed to have zero p-component.

LEMMA 3.1. *Let K be a subgroup of a group G. Then, G/K is p-divisible if and only if K contains a p-basic subgroup of G.*

Proof. Clearly if K contains a p-basic subgroup of G then G/K is p-divisible. Conversely suppose G/K is p-divisible, then $K+pG=G$ therefore $(K+pG)/pG=G/pG$ and the set $\{K+pG\}_{k\in K}$ contains a basis $\{k_\lambda+pG\}_{\lambda\in\Lambda}$, $k_\lambda \in K$, of G/pG. Now since $G_p=0$ the corollary to Proposition 1.3 shows that $B=\bigoplus\langle k_\lambda\rangle$ is a p-basic subgroup of G.

Remark. Lemma 3.1 is false in the case of p-groups. In that case the above proof does not go through as one can only prove that B is p-neat and dense but not necessarily p-pure.

THEOREM 3.2. *Let A be a subgroup of a p-torsion free group G. Then A is quasi-p-large in G if and only if G contains p^nG for some $n \in N$.*

Proof. Since $G_p=0$, there exists a smallest p-pure subgroup K of G containing A. Let B_1 be a p-basic subgroup of K then we can extend B_1 to B a p-basic subgroup of G by a direct summand B_2. That is to say: $B=B_1\oplus B_2$ is a p-basic subgroup of G. Suppose now that A is quasi-p-large in G. Then $G=A+B=K+B=K\oplus B_2$. Now, from Lemma 2.3, for every $q\neq p$, $q\in P$, qB is also a p-basic subgroup of G. It follows that $G=A+qB=K\oplus qB_2$, in other words: $B_2=qB_2$. But B is a free group therefore $B_2=0$, and $K=G$. This means that G/A is a p-primary group since $K/A=(G/A)_p$. It remains to show that G/A is bounded. If not then there would exist H/A in G/A such that $(G/A)/(H/A)\simeq Z(p^\infty)$. This implies G/H is p-divisible and by Lemma 3.1, H contains a p-basic subgroup B of G. Therefore $H\supset A+B=G$ which is a contradiction. It follows that G/A is a bounded p-group and there exists $n\in N$ such that $p^nG\subset A$. The converse is obvious.

COROLLARY 3.3. *Let G be a p-torsion free group then a subgroup A of G is quasi-p-large if and only if it contains a p-large subgroup.*

This property is shared with the class of p-groups. But a quasi-p-large subgroup of a mixed group G does not have to contain a p-large subgroup of G.

4. Some results on the mixed case

We will consider here the so called "p-torsion" case. A group G is said to be *p-torsion* if G/G_p is p-divisible.

LEMMA 4.1. *A group G is p-torsion if and only if it has a torsion p-basic subgroup.*

LEMMA 4.2. *Let A be a subgroup of a p-torsion group G. Then A is a quasi-p-large subgroup of G if and only if both:*

1) *A_p is quasi-p-large in G_p;*
2) *$G/A_p = G_p/A_p \oplus A/A_p$.*

Proof. Clearly if A is quasi-p-large the two conditions are satisfied. Conversely suppose (1) and (2) are satisfied and let B be p-basic subgroup of G then since $B = B_p$ is basic in G_p, $B_p + A_p = G_p$ and from condition 2) $A + G_p = G$. Therefore $B + A = G$ and A is quasi-p-large.

THEOREM 4.3. *The set of quasi-p-large subgroups of a p-torsion reduced group G is closed under finite intersection if and only if G is p-primary or G_p is bounded.*

Proof. If G is p-primary this is a well known property of large and quasi-large subgroups (see [1] and [2]). If G is not p-primary and G_p is not bounded, since G is reduced, there exists L a large subgroup of G_p such that G_p/L is an unbounded direct sum of cyclic groups ([2], Prop. 67.4, p. 13) and therefore there exists a subgroup H of G_p containing L such that $G_p/H \cong Z(p^\infty)$. Therefore $G/H = G_p/H \oplus R/H$ for all subgroups R/H of G/H which are G_p/H-high. From Lemma 4.2 R is a quasi-p-large subgroup of G. Now by a standard technique we can construct M/H another G_p/H-high subgroup of G/H such that $M \neq R$. Now R is quasi-p-large and clearly $M \cap R$ can not be quasi-p-large. Therefore G_p must be bounded. Conversely if G is a p-torsion group and G_p is bounded then $G = G_p \oplus H$ and H is p-divisible. But in this case H is quasi-p-large and all quasi-p-large subgroups must contain H.

Finally we give a necessary condition for a subgroup to be p-large in a p-torsion group.

PROPOSITION 4.4. *Let A be a subgroup of a p-torsion reduced group G such that G_p is not bounded. If A is p-large in G then there exists an Ulm-Kaplansky sequence* u *such that $A = G(\mathrm{u})$.*

Proof. Let A be p-large in G. Then A_p is p-large in G_p. Therefore $A_p = G_p(\mathrm{u})$ for some Ulm-Kaplansky sequence u (see [2], Th. 67.2, p. 11). Say $\mathrm{u} = (n_i)$ where n_i are non-negative integers. From Lemma 4.2 $G/A_p = A/A_p \oplus G_p/A_p$. Now A/A_p is p-divisible. It is easy to check that $A \subset G(\mathrm{u})$ and clearly $G_p(\mathrm{u}) = (G(\mathrm{u}))_p = G(\mathrm{u}) \cap G_p$. Therefore $G(\mathrm{u}) = A$.

In view of Lemma 4.2 and Propositon 4.4 the problem of characterizing the p-large subgroup of the p-torsion case is reduced to finding for given groups G the Ulm-

Kaplansky sequences u for which $G/G(\mathrm{u})_p = G_p/G(\mathrm{u})_p \oplus G(\mathrm{u})/G(\mathrm{u})_p$. In other words $G(\mathrm{u})/G(\mathrm{u})_p$ is a summand of $G/G(\mathrm{u})_p$. The answer is not known even in the case of $\hat{B}$ the p-adic completion of a direct sum of cyclic groups $B = \bigoplus \langle x_n \rangle$, $o(x_n) = p^n$, $n \in N$.

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